

CHARACTERIZATIONS OF ALL-DERIVABLE POINTS IN NEST ALGEBRAS

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ABSTRACT. Let \mathcal{A} be an operator algebra on a Hilbert space. We say that an element $G \in \mathcal{A}$ is an all-derivable point of \mathcal{A} if every derivable linear mapping φ at G (i.e. $\varphi(ST) = \varphi(S)T + S\varphi(T)$ for any $S, T \in \text{alg}\mathcal{N}$ with $ST = G$) is a derivation. Suppose that \mathcal{N} is a nontrivial complete nest on a Hilbert space H . We show in this paper that $G \in \text{alg}\mathcal{N}$ is an all-derivable point if and only if $G \neq 0$.

1. 1. INTRODUCTION AND PRELIMINARIES

Let \mathcal{H} and \mathcal{K} be two Hilbert spaces. $B(H, K)$ stands for the set of all bounded linear operators from H into K , and is abbreviated to $B(H, H)$ to $B(H)$. Let \mathcal{A} be an operator subalgebra in $B(H)$. We say that a linear mapping φ from \mathcal{A} into itself is a derivable mapping at G if $\varphi(ST) = \varphi(S)T + S\varphi(T)$ for any $S, T \in \mathcal{A}$ with $ST = G$. We say that an element $G \in \mathcal{A}$ is an all-derivable point of \mathcal{A} if every derivable linear mapping φ at G is a derivation. Let N be a closed subspace in H . We use the symbols I_N to denote the unit operator in $B(N)$. If \mathcal{N} is a complete nest on H , then the nest algebra $\text{alg}\mathcal{N}$ is the set of all operators which leave every member of \mathcal{N} invariant.

As well known, derivations are very important maps both in theory and applications. In general there are two directions in the study of the local actions of derivations of operator algebras. One is the well known local derivation problem (see [9]). The other is to study conditions under which derivations of operator algebras can be completely determined by the action on some sets of operators. It is obvious that a liner map is a derivation if and only if it is derivable at all points. It is natural and interesting to ask the question whether or not a linear map is a derivation if it is derivable only at one given point.

We describe some of the results related to ours. Jing and Lu [7] showed that every derivable mapping φ at 0 with $\varphi(I) = 0$ on nest algebras is a derivation. Hou and Qi [6] got that every idempotent with the range in \mathcal{N} is an all-derivations in $\text{alg}\mathcal{N}$, where \mathcal{N} is a complete nest on Banach space. Li, Pan and Xu [11] proved that every derivable mapping φ at 0 with $\varphi(I) = 0$ on CSL algebras is a derivation. Zhu and Xiong in [16] showed that every element $G \in \mathcal{TM}_n$ is an all-derivable point of \mathcal{TM}_n if and only if $G \neq 0$, where \mathcal{TM}_n is the algebra of all $n \times n$ upper triangular matrices. For other relative reference, see [1-5,8-10,12-15].

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It is the aim of this paper to prove that an operator $G \in \text{alg}\mathcal{N}$ is an all-derivable point of the nest algebra $\text{alg}\mathcal{N}$ if and only if $G \neq 0$, where \mathcal{N} be a nontrivial complete nest on a Hilbert space H (i.e. there exists an element $N \in \mathcal{N}$ with $N \neq \{0\}, H$).

2. 2. ALL-DERIVABLE POINTS IN NEST ALGEBRAS

Suppose that N is a closed subspace in H . In this section, we always use the symbols I_N to denote the unit operator on N . We easily prove the following two lemmas.

Lemma 2.1 *Let \mathcal{N} be a complete nest on a Hilbert space H . Then G is an all-derivation point of $\text{alg}\mathcal{N}$ if and only if λG is of $\text{alg}\mathcal{N}$ for any real number $\lambda \neq 0$.*

Proof. φ is a derivable mapping at $G \Leftrightarrow \varphi(ST) = \varphi(S)T + S\varphi(T)$ for any $S, T \in \mathcal{A}$ with $ST = G \Leftrightarrow \varphi(\lambda ST) = \varphi(\lambda S)T + \lambda S\varphi(T)$ for any $S, T \in \mathcal{A}$ with $ST = G \Leftrightarrow \varphi(ST) = \varphi(S)T + S\varphi(T)$ for any $S, T \in \mathcal{A}$ with $ST = \lambda G$. \square

Lemma 2.2 *Let \mathcal{A} be an operator subalgebra with unit operator I in $B(H)$, and let φ is a linear mapping from \mathcal{A} into itself. Suppose that $\varphi(X) = 0$ for any invertible operator $X \in \mathcal{A}$, Then $\varphi \equiv 0$.*

Proof. For arbitrary $X \in \text{alg}\mathcal{N}$, there exists a real number $\lambda > \|X\|$. Then both $\lambda I - X$ and $2\lambda I - X$ are two invertible operators. So $\varphi(\lambda I - X) = 0$ and $\varphi(2\lambda I - X) = 0$. It follows from the linearity of φ that $\varphi(X) = 0$. \square

The following is our main theorem in this paper.

Theorem 2.3 *Let \mathcal{N} be a nontrivial complete nest on a Hilbert space H . Then an operator $G \in \text{alg}\mathcal{N}$ is an all-derivable point if and only if $G \neq 0$.*

Proof. Suppose that φ is a derivable linear mapping at $G \neq 0$ from $\text{alg}\mathcal{N}$ into itself. We only need to prove that φ is a derivation. Let $N \in \mathcal{N}$ with $\{0\} \subset N \subset H$. Then all 2×2 operator matrices always are represented as relative to the orthogonal decomposition $H = N \oplus N^\perp$ in the proof of this theorem. Thus we may write

$$G = \begin{bmatrix} D & E \\ 0 & F \end{bmatrix}$$

where $D \in \text{alg}\mathcal{N}_N$, $E \in \text{alg}\mathcal{N}_{N^\perp}$ and $F \in B(N^\perp, N)$ ($\mathcal{N}_N = \{M \cap N : M \in \mathcal{N}\}$ and $\mathcal{N}_{N^\perp} = \{M \cap N^\perp : M \in \mathcal{N}\}$). Without loss of generality, we may assume that $\|D\| < 1$ by Lemma 2.1. For arbitrary $X \in \text{alg}\mathcal{N}_N$, $Y \in B(N^\perp, N)$ and $Z \in \text{alg}\mathcal{N}_{N^\perp}$, we write

$$\begin{cases} \varphi\left(\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} A_{11}(X) & A_{12}(X) \\ 0 & A_{22}(X) \end{bmatrix}, \\ \varphi\left(\begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} B_{11}(Y) & B_{12}(Y) \\ 0 & B_{22}(Y) \end{bmatrix}, \\ \varphi\left(\begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix}\right) = \begin{bmatrix} C_{11}(Z) & C_{12}(Z) \\ 0 & C_{22}(Z) \end{bmatrix}. \end{cases}$$

Obviously, A_{ij}, B_{ij} and C_{ij} ($i, j = 1, 2, i \leq j$) are linear mappings on $\text{alg}\mathcal{N}_N$, $B(N^\perp, N)$ and $\text{alg}\mathcal{N}_{N^\perp}$, respectively.

Step 1. We show that $A_{11}(\cdot)$ is a derivable mapping at D . For arbitrary $X, U \in \text{alg}\mathcal{N}$ with $XU = D$, taking $S = \begin{bmatrix} \lambda^{-1}X & \lambda E \\ 0 & \lambda F \end{bmatrix}, T = \begin{bmatrix} \lambda U & 0 \\ 0 & \lambda^{-1}I_{N^\perp} \end{bmatrix} \in \text{alg}\mathcal{N}$

for any real number $\lambda > 0$, then $ST = G$. It follows that

$$\begin{aligned}
 (2.1) \quad & \begin{bmatrix} A_{11}(D) + B_{11}(E) + C_{11}(F) & A_{12}(D) + B_{12}(E) + C_{12}(F) \\ 0 & A_{22}(D) + B_{22}(E) + C_{22}(F) \end{bmatrix} \\
 &= \varphi(G) = \varphi(S)T + S\varphi(T) \\
 &= \begin{bmatrix} \lambda^{-1}A_{11}(X) + \lambda B_{11}(E) + \lambda C_{11}(F) & * \\ 0 & * \end{bmatrix} \begin{bmatrix} \lambda U & 0 \\ 0 & \lambda^{-1}I_{N^\perp} \end{bmatrix} \\
 &= \begin{bmatrix} \lambda^{-1}X & \lambda E \\ 0 & \lambda F \end{bmatrix} \begin{bmatrix} \lambda A_{11}(U) + \lambda^{-1}C_{11}(I) & * \\ 0 & * \end{bmatrix} \\
 &= \begin{bmatrix} A_{11}(X)U + \lambda^2 B_{11}(E)U + \lambda^2 C_{11}(F)U & * \\ +XA_{11}(U) + \lambda^{-2}XC_{11}(I) & * \\ 0 & * \end{bmatrix}.
 \end{aligned}$$

It follows from the matrix equation that

$$A_{11}(D) + B_{11}(E) + C_{11}(F) = A_{11}(X)U + \lambda^2 B_{11}(E)U + \lambda^2 C_{11}(F)U + XA_{11}(U) + \lambda^{-2}XC_{11}(I)$$

The above equation implies that

$$A_{11}(D) + B_{11}(E) + C_{11}(F) = A_{11}(X)U + XA_{11}(U);$$

$$B_{11}(E)U + C_{11}(F)U = 0; XC_{11}(I) = 0.$$

Furthermore $C_{11}(I) = 0$ and $B_{11}(E) + C_{11}(F) = 0$. Thus $A_{11}(D) = A_{11}(X)U + XA_{11}(U)$, i.e. $A_{11}(\cdot)$ is a derivable mapping at D .

Step 2. We shows that $C_{11}(W) = 0$ and $B_{11}(V) = 0$ for any $W \in \text{alg}\mathcal{N}_{N^\perp}$ and $V \in B(N^\perp, N)$.

Letting $S = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$, $T = \begin{bmatrix} U & V \\ 0 & W \end{bmatrix} \in \text{alg}\mathcal{N}$ with $ST = G$, then $XU = D$, $XV + YW = E$ and $ZW = F$. Since φ is a derivable mapping at G on $\text{alg}\mathcal{N}$, we have

$$\begin{aligned}
 (2.2) \quad & \begin{bmatrix} A_{11}(D) + B_{11}(E) + C_{11}(F) & A_{12}(D) + B_{12}(E) + C_{12}(F) \\ 0 & A_{22}(D) + B_{22}(E) + C_{22}(F) \end{bmatrix} \\
 &= \varphi(G) = \varphi(S)T + S\varphi(T) \\
 &= \begin{bmatrix} A_{11}(X) + B_{11}(Y) & A_{12}(X) + B_{12}(Y) \\ +C_{11}(Z) & +C_{12}(Z) \\ 0 & A_{22}(X) + B_{22}(Y) + C_{22}(Z) \end{bmatrix} \begin{bmatrix} U & V \\ 0 & W \end{bmatrix} \\
 &= \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \begin{bmatrix} A_{11}(U) + B_{11}(V) & A_{12}(U) + B_{12}(V) \\ +C_{11}(W) & +C_{12}(W) \\ 0 & A_{22}(U) + B_{22}(V) + C_{22}(W) \end{bmatrix}.
 \end{aligned}$$

The above equation implies the following three equations hold.

$$\begin{aligned}
 (2.3) \quad & A_{11}(D) + B_{11}(E) + C_{11}(F) \\
 &= A_{11}(X)U + B_{11}(Y)U + C_{11}(Z)U \\
 &\quad + XA_{11}(U) + XB_{11}(V) + XC_{11}(W);
 \end{aligned}$$

$$\begin{aligned}
(2.4) \quad & A_{12}(D) + B_{12}(E) + C_{12}(F) \\
= & A_{11}(X)V + B_{11}(Y)V + C_{11}(Z)V \\
& + A_{12}(X)W + B_{12}(Y)W + C_{12}(Z)W \\
& + XA_{12}(U) + XB_{12}(V) + XC_{12}(W) \\
& + YA_{22}(U) + YB_{22}(V) + YC_{22}(W);
\end{aligned}$$

$$\begin{aligned}
(2.5) \quad & A_{22}(D) + B_{22}(E) + C_{22}(F) \\
= & A_{22}(X)W + B_{22}(Y)W + C_{22}(Z)W \\
& + ZA_{22}(U) + ZB_{22}(V) + ZC_{22}(W);
\end{aligned}$$

Note that A_{11} is a derivable mapping at D . So we have $A_{11}(XU) = A_{11}(D) = A_{11}(X)U + XA_{11}(U)$ for any $X, U \in \text{alg}\mathcal{N}_N$. By Eq. (2.3), we have

$$(2.6) \quad B_{11}(E) + C_{11}(F) = B_{11}(Y)U + C_{11}(Z)U + XB_{11}(V) + XC_{11}(W).$$

Taking $X = D, U = I_N, Y = E, V = 0, Z = F$ and $W = I_{N^\perp}$ in Eq. (2.6), then $DC_{11}(I_{N^\perp}) = 0$. Letting $X = D, U = I_N, Y = -E, V = 0, Z = -F$ and $W = -I_{N^\perp}$ in Eq. (2.6), then $B_{11}(E) + C_{11}(F) = 0$. It follows that

$$0 = B_{11}(Y)U + C_{11}(Z)U + XB_{11}(V) + XC_{11}(W).$$

For arbitrary $V \in B(N^\perp, N)$, if we put $X = I_N, U = D, W = I_{N^\perp}, Z = F$ and $Y = E - V$ in the above equation, then $B_{11}(V)(I - D) = 0$. Since $\|D\| < 1$ implies that $I - D$ is invertible, $B_{11}(V) = 0$. For arbitrary $W \in \text{alg}\mathcal{N}_{N^\perp}$, if we put $X = I_N, Y = 0, Z = FW^{-1}, U = D$ and $V = E$ in the above equation, then

$$C_{11}(FW^{-1})D + C_{11}(W) = 0.$$

Replacing W by $2W$ in the above equation, we have

$$\frac{1}{2}C_{11}(FW^{-1})D + 2C_{11}(W) = 0.$$

The above two equations implies that $C_{11}(W) = 0$ for any invertible operator $W \in \text{alg}\mathcal{N}_{N^\perp}$. By Lemma 2.2, $C_{11}(W) \equiv 0$.

Step 3. We show that $A_{22}(X) = 0$ and $B_{22}(Y) = 0$ for any $X \in \text{alg}\mathcal{N}_N$ and $Y \in B(N^\perp, N)$.

Letting $X = D, U = I_N, Y = E, V = 0, Z = F$ and $W = I_{N^\perp}$ in Eq. (2.5), then

$$(2.7) \quad F(A_{22}(I_N) + C_{22}(I_{N^\perp})) = 0.$$

On the other hand, for arbitrary $Y \in B(N^\perp, N)$ and real number $\lambda \neq 0$, putting $X = \lambda I_N, U = \lambda^{-1}D, V = Y + \lambda^{-1}E, Z = -\lambda^{-1}F$ and $W = -\lambda I_{N^\perp}$ in Eq. (2.5), then

$$\begin{aligned}
A_{22}(D) + B_{22}(E) &= -\lambda^2 A_{22}(I_N) - \lambda B_{22}(Y) - \lambda^{-2} F A_{22}(D) \\
&- \lambda^{-1} F B_{22}(Y) - \lambda^{-2} F B_{22}(E) + F C_{22}(I_{N^\perp}).
\end{aligned}$$

The above equation implies that

$$(2.8) \quad A_{22}(D) + B_{22}(E) = F C_{22}(I_{N^\perp}); A_{22}(I_N) = 0;$$

$$(2.9) \quad F A_{22}(D) + F B_{22}(E) = 0; B_{22}(Y) = 0.$$

It follows from Eq. (2.7)-(2.9) that $F C_{22}(I_{N^\perp}) = 0$ and

$$A_{22}(D) = F C_{22}(I_{N^\perp}) = 0.$$

Bring Eq. (2.9) and the above equation to Eq. (2.5) to get

$$C_{22}(F) = A_{22}(X)W + C_{22}(Z)W + ZA_{22}(U) + ZC_{22}(W).$$

For any invertible operator $X \in \text{alg}\mathcal{N}_N$, taking $U = X^{-1}D$, $V = 0$, $W = I_{N^\perp}$, $Y = E$ and $Z = F$ in the above equation, then we have

$$A_{22}(X) + FA_{22}(X^{-1}D) = 0.$$

Replacing X by $2X$, we have

$$2A_{22}(X) + \frac{1}{2}FA_{22}(X^{-1}D) = 0.$$

The above two equations implies that $A_{22}(X) = 0$ for any invertible operator $X \in \text{alg}\mathcal{N}_N$. By Lemma 2.2, $A_{22}(X) \equiv 0$.

Step 4. We show that $A_{12}(X) = -XC_{12}(I_{N^\perp})$ and $C_{12}(W) = C_{12}(I_{N^\perp})W$ for any $X \in \text{alg}\mathcal{N}_N$ and $W \in \text{alg}\mathcal{N}_{N^\perp}$. For arbitrary invertible operator $X \in \text{alg}\mathcal{N}_N$, taking $U = X^{-1}D$, $V = 0$, $Y = \lambda E$, $Z = \lambda F$ and $W = \lambda^{-1}I_{N^\perp}$ in Eq. (2.4), then

$$(2.10) \quad \begin{aligned} A_{12}(D) &= \lambda^{-1}A_{12}(X) + XA_{12}(X^{-1}D) \\ &\quad + \lambda^{-1}XC_{12}(I_{N^\perp}) + \lambda EA_{22}(X^{-1}D) + EC_{22}(I_{N^\perp}); \end{aligned}$$

Eq. (2.10) implies that

$$A_{12}(X) = -XC_{12}(I_{N^\perp}).$$

It follows from Lemma 2.2 that $A_{12}(X) = -XC_{12}(I_{N^\perp})$ for any $X \in \text{alg}\mathcal{N}$.

For arbitrary invertible operator $W \in \text{alg}\mathcal{N}_{N^\perp}$, taking $X = I_N$, $U = D$, $Y = 0$, $V = E$ and $Z = FW^{-1}$ in Eq. (2.4), then

$$(2.11) \quad C_{12}(F) = A_{11}(I_N)E + A_{12}(I_N)W + C_{12}(FW^{-1})W + C_{12}(W).$$

Replacing W by $2W$ in the above equation, we have

$$(2.12) \quad C_{12}(F) = A_{11}(I_N)E + 2A_{12}(I_N)W + C_{12}(FW^{-1})W + 2C_{12}(W).$$

Combining Eq. (2.11) with Eq. (2.12), we obtain

$$A_{12}(I_N)W + C_{12}(W) = 0.$$

Letting $W = I_{N^\perp}$ in the above equation, then $A_{12}(I_N) = -C_{12}(I_{N^\perp})$. Hence $C_{12}(W) = C_{12}(I_{N^\perp})W$. It follows from Lemma 2.2 that $C_{12}(W) = C_{12}(I_{N^\perp})W$ for any $W \in \text{alg}\mathcal{N}^\perp$.

Step 5. We show that both $A_{11}(I_N) = 0$ and $C_{22}(I_{N^\perp}) = 0$.

For arbitrary $V \in B(N^{-1}, N)$, taking $X = I_N$, $U = D$, $Y = E - V$, $Z = F$ and $W = I_{N^\perp}$ in Eq. (2.4), by the results of Step 2-3, we have

$$(2.13) \quad A_{11}(I_N)V = VC_{22}(I_{N^\perp}) - EC_{22}(I_{N^\perp}).$$

Letting $V = 0$ in Eq. (2.13), then $EC_{22}(I_{N^\perp}) = 0$. Hence $A_{11}(I_N)V = VC_{22}(I_{N^\perp})$ for any $V \in B(N^{-1}, N)$. For arbitrary $x \in N$ and $y \in N^{-1}$, we have $x \otimes y \in B(N^{-1}, N)$. So $A_{11}(I_N)x \otimes y = x \otimes C_{22}(I_{N^\perp})^*y$. Thus there exists a complex number α such that $A_{11}(I_N) = \alpha I_N$ and $C_{22}(I_{N^\perp}) = \alpha I_{N^\perp}$. Note that $FC_{22}(I_{N^\perp}) = 0$ and $EC_{22}(I_{N^\perp}) = 0$. So $\alpha F = 0$ and $\alpha E = 0$. On the other hand, taking $X = I_N$, $U = D$, $V = E$, $Y = 0$, $Z = I$ and $W = F$ in Eq. (2.3), then $A_{11}(I_N)D = 0$, i.e. $\alpha D = 0$. Since $G \neq 0$, $\alpha = 0$. It follows that $A_{11}(I_N) = 0$ and $C_{22}(I_{N^\perp}) = 0$.

Step 6. We show that both $A_{11}(\cdot)$ and $C_{22}(\cdot)$ are derivations.

For arbitrary $Y \in B(N^\perp, N)$ and invertible operator $W \in \text{alg}\mathcal{N}_{N^\perp}$, taking $X = I_N$, $U = D$, $V = E - YW$ and $Z = FW^{-1}$ in Eq. (2.4), then

$$(2.14) \quad B_{12}(YW) = B_{12}(Y)W + YC_{22}(W).$$

For any $W_1, W_2 \in \text{alg}\mathcal{N}_{N^\perp}$, we have

$$(2.15) \quad B_{12}(YW_1W_2) = B_{12}(Y)W_1W_2 + YC_{22}(W_1W_2).$$

On the other hand,

$$(2.16) \quad \begin{aligned} B_{12}(YW_1W_2) &= B_{12}(YW_1)W_2 + YW_1C_{22}(W_2) \\ &= B_{12}(Y)W_1W_2 + YC_{22}(W_1)W_2 + YW_1C_{22}(W_2). \end{aligned}$$

The Eq. (2.15) and Eq. (2.16) implies that

$$C_{22}(W_1W_2) = C_{22}(W_1)W_2 + W_1C_{22}(W_2).$$

Hence $C_{22}(\cdot)$ is a derivation.

For arbitrary $V \in B(N^\perp, N)$ and invertible operator $X \in \text{alg}\mathcal{N}_N$, taking $X^{-1}D$, $Y = E - XV$, $W = I_{N^\perp}$ and $Z = F$, then

$$(2.17) \quad B_{12}(XV) = A_{11}(X)V + XB_{12}(V).$$

Similarly, we can prove that $A_{11}(\cdot)$ is a derivation.

Step 7. We show that φ is a derivation. For arbitrary $S = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$ and $T = \begin{bmatrix} U & V \\ 0 & W \end{bmatrix}$ in $\text{alg}\mathcal{N}$, we only need to prove that $\varphi(ST) = \varphi(S)T + S\varphi(T)$. By

Eqs. (2.14), (2.17) and the results of Step 2-6, we easily calculate

$$\begin{aligned}
& \varphi(S)T + S\varphi(T) \\
&= \begin{bmatrix} A_{11}(X) & A_{12}(X) + B_{12}(Y) + C_{12}(Z) \\ 0 & C_{22}(Z) \end{bmatrix} \begin{bmatrix} U & V \\ 0 & W \end{bmatrix} \\
&+ \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \begin{bmatrix} A_{11}(U) & A_{12}(U) + B_{12}(V) + C_{12}(W) \\ 0 & C_{22}(W) \end{bmatrix} \\
&= \begin{bmatrix} A_{11}(X)U + XA_{11}(U) & A_{11}(X)V + A_{12}(X)W + B_{12}(Y)W \\ & + C_{12}(Z)W + XA_{12}(U) + XB_{12}(V) \\ & + XC_{12}(W) + YC_{22}(W) \\ 0 & C_{22}(Z)W + ZC_{22}(W) \end{bmatrix} \\
&= \begin{bmatrix} A_{11}(XU) & A_{12}(X)W + B_{12}(YW) + C_{12}(Z)W \\ & + XA_{12}(U) + B_{12}(XV) + XC_{12}(W) \\ 0 & C_{22}(ZW) \end{bmatrix} \\
&= \begin{bmatrix} A_{11}(XU) & A_{12}(X)W + B_{12}(YW + XV) \\ & + C_{12}(Z)W + XA_{12}(U) + XC_{12}(W) \\ 0 & C_{22}(ZW) \end{bmatrix} \\
&= \begin{bmatrix} A_{11}(XU) & -XC_{12}(I_{N^\perp})W + B_{12}(YW + XV) \\ & + C_{12}(I_{N^\perp})ZW - XUC_{12}(I_{N^\perp}) + XC_{12}(I_{N^\perp})W \\ 0 & C_{22}(ZW) \end{bmatrix} \\
&= \begin{bmatrix} A_{11}(XU) & B_{12}(YW + XV) + C_{12}(I_{N^\perp})ZW - XUC_{12}(I_{N^\perp}) \\ 0 & C_{22}(ZW) \end{bmatrix} \\
&= \begin{bmatrix} A_{11}(XU) & B_{12}(YW + XV) + C_{12}(ZW) + A_{12}(XU) \\ 0 & C_{22}(ZW) \end{bmatrix} \\
&= \varphi(ST).
\end{aligned}$$

This completes the proof. \square

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